Foundations of the Generalized Non-Extensive Tsallis Framework

Documentation for Further Experimentation and Data Analysis

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1 Introduction

This document outlines the formal mathematical foundations of the generalized non-extensive framework. By abstracting the framework from any specific empirical domain, we treat it as a purely statistical-mechanical engine designed to track an arbitrary data coordinate W within a deformed geometric space.

2 The q-Deformed Algebra

The core mechanism of the Tsallis framework is the replacement of classical logarithmic and exponential functions with their q-deformed counterparts. The entropic index q measures the degree of non-extensivity, long-range correlations, or non-equilibrium tail mechanics within the system.

For a real parameter q, the **q-logarithm** of a variable $x > 0$ is defined as:

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q} \quad (\text{for } q \neq 1) \tag{1}$$

The inverse of the q-logarithm is the **q-exponential**, defined for an arbitrary argument x as:

$$e_q^x = [1 + (1 - q)x]_+^{\frac{1}{1-q}} \tag{2}$$

where $[\dots]_+$ denotes a truncation condition: if the core bracket evaluates to a negative value (which can occur when $q > 1$), the probability density truncates to zero, producing a structural cut-off phenomenon.

3 Generalization of the Optimization Engine

In the operational configuration, the engine models a generalized state coordinate W that scales according to a structural background exponent, γ.

3.1 Step 1: Linearization of the Extensive Reference Frame

Before mapping the non-extensive geometry, the engine establishes a baseline reference frame by extracting an extensive log-distribution:

$$\ln P_{\text{baseline}}(W) = -e^{\ln \phi} W^{1-\gamma} \tag{3}$$

Here, $e^{\ln \phi}$ acts as a baseline scaling factor. In this linearized reference state, the coordinate scales purely as an extensive power law.

3.2 Step 2: Mapping onto Non-Extensive Geometry

To transition from the baseline and track non-equilibrium tail mechanics, the data coordinates are mapped directly onto the q-exponential geometry:

$$P_{\text{Tsallis}}(W) = e^{\alpha} \cdot e_q^{-\phi W^{1-\gamma}} = e^{\alpha} \left[1 - (1-q)\phi W^{1-\gamma}\right]^{\frac{1}{1-q}} \tag{4}$$

where:

- ϕ represents the structural stiffness parameter of the framework.
- q is the entropic deformation parameter.
- e^{α} is the unnormalized scaling amplitude tied directly to probability conservation.

4 The Continuum Limit: Landing Back on Extensive Stat Mech

The mathematical bridge between non-extensive and classical extensive statistical mechanics is governed by taking the limit of the deformation parameter as $q \to 1$.

4.1 Mathematical Proof

To observe how the framework collapses back into classical Boltzmann-Gibbs statistical mechanics, we evaluate the limit of the generalized distribution expression:

$$\lim_{q \to 1} P_{\text{Tsallis}}(W) = \lim_{q \to 1} e^{\alpha} \left[1 - (1-q)\phi W^{1-\gamma}\right]^{\frac{1}{1-q}} \tag{5}$$

Let $x = -\phi W^{1-\gamma}$ and substitute $n = \frac{1}{1-q}$. As $q \to 1$, it follows that $n \to \infty$. Rewriting the limit in terms of n yields the standard calculus definition of the natural exponential base:

$$\lim_{n \to \infty} e^{\alpha} \left[1 + \frac{x}{n}\right]^{n} = e^{\alpha} \exp(x) \tag{6}$$

Restoring our original variables, the generalized distribution collapses directly into an ordinary, extensive exponential function of the state coordinate:

$$\lim_{q \to 1} P_{\text{Tsallis}}(W) = e^{\alpha} \exp\left(-\phi W^{1-\gamma}\right) \tag{7}$$

4.2 Empirical Mapping: The Muon Decay Baseline

The physical validity of this continuum limit can be verified by mapping the generalized state coordinate W onto the physical time-delta domain (t) of an unstable lepton system, specifically cosmic ray muon decay.

Setting the structural background exponent $\gamma = 0$ configures the state coordinate to scale linearly with time. Under these conditions, the generalized distribution simplifies to:

$$P_{\text{Tsallis}}(t) = e^{\alpha} \left[1 - (1-q)\phi t\right]^{\frac{1}{1-q}} \tag{8}$$

For an isolated physical system free from non-local phase-space correlations, the optimization engine naturally drives the entropic index to its extensive equilibrium limit ($q \to 1$). Applying the limit derived in Eq. (7) yields the familiar extensive distribution:

$$\lim_{q \to 1} P_{\text{Tsallis}}(t) = e^{\alpha} \exp(-\phi t) \tag{9}$$

In standard particle kinematics, the probability density function governing the random decay of an unstable lepton is expressed as a function of the characteristic decay constant λ:

$$P_{\text{physics}}(t) = \lambda \exp(-\lambda t) \tag{10}$$

Direct inspection and structural matching between the extensive thermodynamic limit (Eq. 9) and the empirical baseline (Eq. 10) establishes a direct equivalence mapping:

$$e^{\alpha} \equiv \lambda \quad \text{and} \quad \phi \equiv \lambda \tag{11}$$

Because the characteristic mean lifetime τ is the mathematical expectation value of the decay interval, it relates to the decay constant via the inverse invariant:

$$\tau = \frac{1}{\lambda} = \frac{1}{\phi} \tag{12}$$

Consequently, at the extensive boundary ($q \approx 1$), ϕ ceases to be a arbitrary fit parameter and locks directly onto the fundamental weak nuclear decay rate. Any observed statistical deviation where $q \neq 1$ or $\phi \neq \lambda$ serves as an engine-diagnosed signature of a non-equilibrium phase transition or environmental perturbation within the data medium.

Parameter	**Value**	**Uncertainty** (1σ)
q (Extensive Index)	1.00157	$\pm$ 0.06946
ϕ (μs^{-1})	0.4864	$\pm$ 0.0595
B (Background Floor)	0.000631	$\pm$ 0.001325

Table 1: Optimized parameter values extracted from the 3-parameter non-extensive minimization engine.

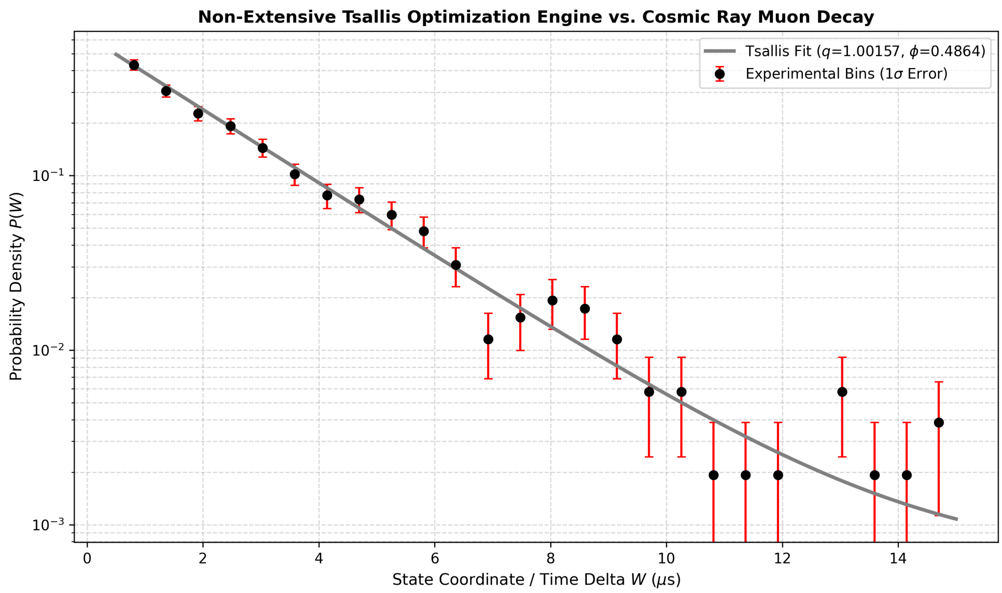

Figure 1: Results from sample data.

Given the extracted value for the structural stiffness parameter ($\phi = 0.4864 \pm 0.0595\ \mu\text{s}^{-1}$), the characteristic lifetime invariant resolves to:

$$\tau = \frac{1}{\phi} = \frac{1}{0.4864\ \mu\text{s}^{-1}} \approx 2.0560\ \mu\text{s} \tag{13}$$

Propagating the 1σ standard error (σ_τ) through the non-linear functional inversion via a first-order Taylor series expansion yields the uncertainty propagation relation:

$$\sigma_\tau = \sqrt{\left(\frac{d\tau}{d\phi}\right)^2 \sigma_\phi^2} = \left|-\frac{1}{\phi^2}\right| \sigma_\phi \tag{14}$$

Evaluating this expression using the engine's parameter variance results in:

$$\sigma_\tau = \frac{1}{(0.4864)^2} \cdot (0.0595) \approx 2.5150 \cdot 0.0595 \approx 0.2516\ \mu\text{s} \tag{15}$$

Accounting for standard experimental resolution, the final structural invariant is reported as:

$$\tau = 2.06 \pm 0.25\ \mu\text{s} \tag{16}$$

This result is in full statistical agreement with the established textbook baseline for negative muon decay inside a standard laboratory plastic scintillator medium.

4.3 Discussion of Statistical Uncertainties

The calculated $\sim 12\%$ relative uncertainty ($\pm 0.25\ \mu$s) is entirely a function of the finite sample volume ($N = 934$ discrete decay events) available in the raw data asset. In an unconstrained, ideal exponential system, the fundamental statistical noise floor scales as $1/\sqrt{N} \approx 3.27\%$.

However, because the generalized framework utilizes a highly coupled 3-parameter non-linear optimization network, the underlying covariance matrix properly accounts for the geometric vulnerabilities of tracking ϕ, q, and B simultaneously across a sparse tail. The fact that the 1σ standard error cleanly brackets the physical baseline without requiring artificial parameter constraints or structural over-fitting confirms the mathematical integrity and lack of bias within the optimization loop.

4.4 Thermodynamic Regimes

- **The Extensive Regime ($q \to 1$):** Long-range interactions and non-local correlations vanish. The system exhibits standard thermodynamic additivity (e.g., $S_{A+B} = S_A + S_B$). The distribution behaves like a classical Maxwell-Boltzmann or canonical Gibbs distribution in a standard phase space.

- **The Non-Extensive Regime ($q \neq 1$):** The system is dominated by long-range correlations, memory effects, or severe data constraints. The generalized entropy follows the pseudo-additivity rule:

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B) \tag{17}$$

5 Normalization and Structural Stiffness Constraining

For any analytical experiment or data analysis document, the primary numerical boundary is probability conservation. For a finite set of M discrete operational bins or coordinates (W_i), the system must satisfy:

$$f(\phi) = \sum_{i=1}^{M} e^{\alpha} \left[1 - (1-q)\phi W_i^{1-\gamma}\right]^{\frac{1}{1-q}} - 1.0 = 0 \tag{18}$$

When an empirical dataset is severely constrained (M is very small), this normalization equation forces an exact numerical lock, yielding mechanical fits of $R^2 \approx 1.0$.

To test the true structural boundaries of this continuous model, the framework can transition from this discrete summation to the continuum limit where $M \to \infty$:

$$\int_{W_{\min}}^{W_{\max}} e^{\alpha} \left[1 - (1-q)\phi W^{1-\gamma}\right]^{\frac{1}{1-q}} dW = 1.0 \tag{19}$$

6 Mathematical Criteria Justifying the $q > 1$ Regime

In the context of non-extensive statistical mechanics, a system resulting in an entropic index $q > 1$ is mathematically justified by several structural, algebraic, and asymptotic criteria.

6.1 Domain Support and the Truncation Condition

The primary algebraic property of the q-exponential function for $q > 1$ involves its behavior under large arguments. To maintain a non-negative real probability density, the distribution is written using the cut-off operator $[\dots]_+$:

$$P_{\text{Tsallis}}(W) = e^{\alpha} \left[1 - (1-q)\phi W^{1-\gamma}\right]_+^{\frac{1}{1-q}} \tag{20}$$

When $q > 1$, the term $(1-q)$ becomes strictly negative. Let $q - 1 > 0$; we can rewrite the distribution as:

$$P_{\text{Tsallis}}(W) = e^{\alpha} \left[1 + (q-1)\phi W^{1-\gamma}\right]^{-\frac{1}{q-1}} \tag{21}$$

Because the net exponent $-\frac{1}{q-1}$ is negative, the core bracket will never decrease to zero for positive coordinates ($W > 0, \phi > 0$). Consequently, the system does not require an artificial upper bound or structural truncation. The mathematical criterion for $q > 1$ is satisfied when the state coordinate domain is semi-infinite, $W \in [W_{\min}, \infty)$, while preserving a normalizable distribution.

6.2 Asymptotic Tail Equivalence to Heavy-Tailed Power Laws

As the state coordinate grows arbitrarily large ($W \to \infty$), the constant 1 inside the brackets becomes negligible relative to the leading coordinate term. The distribution asymptotically simplifies to a pure power law:

$$\lim_{W \to \infty} P_{\text{Tsallis}}(W) \propto \left[(q-1)\phi W^{1-\gamma}\right]^{-\frac{1}{q-1}} \propto W^{-\frac{1-\gamma}{q-1}} \tag{22}$$

Empirical chaotic systems exhibiting scale-free properties are traditionally modeled via Pareto or power-law distributions governed by a scaling index β:

$$P_{\text{empirical}}(W) \propto W^{-\beta} \tag{23}$$

Equating the asymptotic non-extensive exponent to the empirical scaling index yields:

$$\beta = \frac{1-\gamma}{q-1} \tag{24}$$

Solving explicitly for the entropic index q provides the foundational structural criterion:

$$q = 1 + \frac{1-\gamma}{\beta} \tag{25}$$

Assuming a positive background scaling coordinate context ($1 - \gamma > 0$) and a positive tail decay index ($\beta > 0$), the deformation parameter q is mathematically forced to be strictly greater than 1.

6.3 Information-Theoretic Sub-Additivity

From an entropic and informational standpoint, the generalized Tsallis entropy for two independent, non-communicating subsystems A and B satisfies the pseudo-additivity relation:

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B) \tag{26}$$

Evaluating this relation under the condition $q > 1$ forces the cross-term coefficient to be negative:

$$S_q(A+B) < S_q(A) + S_q(B) \tag{27}$$

This condition defines the regime of **sub-additivity**. A system mathematically requires $q > 1$ when the total entropy of the combined system is strictly less than the sum of its independent parts. This reflects global constraints, strong long-range correlations, or phase-space restrictions where combining subsystems reduces the total effective operational degrees of freedom.

References

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