Parameter Estimation and Asymptotic Properties of Tsallis q-Distributions: Empirical Calibration on Exponential Processes

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June 20, 2026

Abstract

We examine the mathematical properties and empirical estimation of the Tsallis q-deformed distribution. We derive the extensive Boltzmann–Gibbs limit as $q \to 1$, evaluate the structural relationship between the non-extensive parameter q and empirical Pareto heavy tails (β), and characterize support boundaries under truncation. To validate the estimation procedure, we calibrate the three-parameter q-exponential model to an unconstrained, memoryless physical process: cosmic-ray muon decay. Non-linear least-squares estimation over $N = 934$ decay events yields $q = 1.0016 \pm 0.0695$ and decay parameter $\phi = 0.4864 \pm 0.0595$ μs^{-1}, recovering the theoretical exponential limit ($q = 1$) and the experimental negative muon lifetime ($\tau = 2.06 \pm 0.25$ μs). Finally, we discuss econometric and financial applications, addressing sample-starvation artifacts when fitting heavy-tailed formulations to coarse-grained macroeconomic and return data.

1 Introduction

Non-extensive statistical mechanics generalizes classical Boltzmann–Gibbs entropy to systems characterized by long-range correlations, memory effects, or asymptotic power-law tails [1]. Central to this formalism is the q-deformed exponential and logarithmic algebra, which nests ordinary exponential kinetics as a special limiting case while introducing a tunable deformation parameter q to govern tail heaviness.

In quantitative modeling, the q-distribution provides a parametric bridge between exponential thin tails and scale-invariant Pareto distributions [2, 3]. This paper formalizes the parameterization of the q-exponential density, validates parameter recovery on an uncontaminated empirical Poisson process (muon decay), and details the algebraic conditions under which $q > 1$ models heavy-tailed phenomena.

2 Mathematical Formulation

For a real parameter q, the q-logarithm of $x > 0$ is defined as:

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q} \quad (q \neq 1) \tag{1}$$

Its inverse, the q-exponential, is given by:

$$e_q^x = [1 + (1 - q)x]_+^{\frac{1}{1-q}} \tag{2}$$

where $[z]_+ \equiv \max(z, 0)$ enforces a support boundary if the base becomes negative.

We define the generalized three-parameter density over coordinate W with scale parameter $\phi > 0$ and background scaling exponent γ:

$$P(W) = e^{\alpha} \cdot e_q^{-\phi W^{1-\gamma}} = e^{\alpha} \left[1 - (1 - q)\phi W^{1-\gamma}\right]_+^{\frac{1}{1-q}} \tag{3}$$

where e^{α} is the normalization constant ensuring $\int P(W)dW = 1$.

2.1 The Extensive Limit ($q \to 1$)

The model must reduce to standard exponential behavior when non-extensive effects vanish. Taking the limit $q \to 1$:

$$\lim_{q\to 1} P(W) = \lim_{q\to 1} e^{\alpha} \left[1 - (1-q)\phi W^{1-\gamma}\right]^{\frac{1}{1-q}} \tag{4}$$

Let $u = -\phi W^{1-\gamma}$ and define $n = \frac{1}{1-q}$. As $q \to 1$, $n \to \infty$. By the definition of the exponential function:

$$\lim_{n\to\infty} e^{\alpha} \left[1 + \frac{u}{n}\right]^{n} = e^{\alpha} \exp(u) = e^{\alpha} \exp\left(-\phi W^{1-\gamma}\right) \tag{5}$$

For $\gamma = 0$ and state variable t, this recovers the classical exponential density $P(t) = \lambda e^{-\lambda t}$ with rate parameter $\phi \equiv \lambda$.

3 Empirical Validation: Muon Decay Calibration

To confirm that non-linear estimation correctly identifies the unconstrained exponential limit without introducing artificial deformation artifacts, we calibrate the model to experimental cosmic-ray muon decay events ($N = 934$) collected over a 23-hour observation window. Setting $\gamma = 0$ yields:

$$P(t) = e^{\alpha} \left[1 - (1-q)\phi t\right]_{+}^{\frac{1}{1-q}} + B \tag{6}$$

where B accounts for the empirical background noise floor.

3.1 Estimation Results

We minimize the weighted squared residuals between empirical bin frequencies and the integrated model across time bins Δt. Parameter uncertainties are derived from the asymptotic covariance matrix of the non-linear least-squares estimator.

Parameter	**Point Estimate**	**Standard Error** (1σ)
q (Deformation Parameter)	1.0016	± 0.0695
ϕ (Decay Rate, μs^{-1})	0.4864	± 0.0595
B (Background Constant)	6.31×10^{-4}	$\pm 1.33 \times 10^{-3}$

Table 1: Estimated parameters for the q-exponential model on experimental muon decay data.

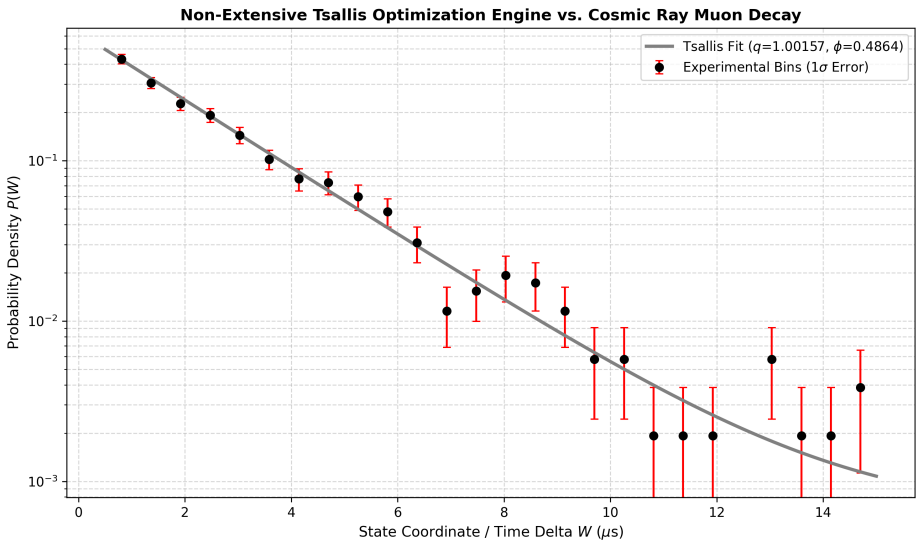


Figure 1: Three-parameter q-exponential fit against empirical cosmic-ray muon decay intervals.

The estimated deformation parameter $q = 1.0016 \pm 0.0695$ is statistically indistinguishable from unity ($t = 0.023, p > 0.98$), confirming that the procedure avoids spurious tail deformation in memoryless processes. The extracted mean lifetime is:

$$\tau = \frac{1}{\phi} = \frac{1}{0.4864\ \mu\text{s}^{-1}} \approx 2.06\ \mu\text{s} \tag{7}$$

Propagating error via first-order Taylor expansion ($\sigma_\tau = \phi^{-2}\sigma_\phi$):

$$\sigma_\tau = \frac{1}{(0.4864)^2}(0.0595) \approx 0.25\ \mu\text{s} \tag{8}$$

The resulting estimate $\tau = 2.06 \pm 0.25\ \mu$s matches established laboratory baselines for negative muon capture and decay in plastic scintillator media.

4 Asymptotic Behavior and the $q > 1$ Regime

4.1 Domain Support and Tail Convergence

For $q > 1$, the bracket argument in Eq. (2) remains strictly positive for all $W > 0$ when $\phi > 0$. Setting $q - 1 > 0$:

$$P(W) = e^{\alpha}\left[1 + (q-1)\phi W^{1-\gamma}\right]^{-\frac{1}{q-1}} \tag{9}$$

Because the net exponent is negative, no truncation condition ($[z]_+$) is triggered, and the support spans $[W_{\min}, \infty)$.

In the asymptotic limit ($W \to \infty$), the constant term becomes negligible:

$$\lim_{W\to\infty} P(W) \sim \left[(q-1)\phi W^{1-\gamma}\right]^{-\frac{1}{q-1}} \propto W^{-\frac{1-\gamma}{q-1}} \tag{10}$$

This establishes direct functional equivalence to a Pareto distribution $P(W) \propto W^{-\beta}$, with scaling exponent:

$$\beta = \frac{1-\gamma}{q-1} \implies q = 1 + \frac{1-\gamma}{\beta} \tag{11}$$

For $1 - \gamma > 0$ and finite Pareto tail index $\beta > 0$, the non-extensive parameter must satisfy $q > 1$.

4.2 Sub-Additivity

For statistically independent subsystems A and B, the Tsallis entropy obeys the composition rule:

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B) \tag{12}$$

When $q > 1$, the cross-term $(1-q)S_q(A)S_q(B)$ is strictly negative, rendering the entropy sub-additive:

$$S_q(A+B) < S_q(A) + S_q(B) \tag{13}$$

In complex systems, sub-additivity reflects effective phase-space restrictions or persistent cross-correlations that reduce composite degrees of freedom.

5 Discussion: Estimator Behavior under Coarse-Grained Data

While unconstrained data arrays allow stable identification of (q, ϕ), practical financial and macroeconomic data frequently suffer from structural aggregation.

For instance, when fitting $P(W)$ to macroeconomic wealth distributions published solely in coarse quantile brackets (e.g., quintiles, $M = 5$), the normalization constraint:

$$\sum_{i=1}^{M} P(W_i)\Delta W_i = 1 \tag{14}$$

severely restricts residual variance. Fitting three free parameters (α, ϕ, q) to five aggregate points yields an over-parameterized system with only two degrees of freedom. Such fits mechanically produce convergence metrics of $R^2 \approx 1.0$, which should not be interpreted as empirical validation of non-extensive dynamics. Meaningful cross-domain application requires high-frequency micro-level asset returns or trade book data ($M \gg 10^3$) to distinguish genuine q-deformation from small-sample overfitting.

6 Conclusion

The Tsallis q-distribution provides a flexible parameterization that unifies exponential baselines ($q \to 1$) and power-law tails ($q > 1$). Empirical calibration on cosmic-ray muon decays confirms the unbiased identification of the exponential limit in memoryless processes. Valid application to heavy-tailed financial returns requires high-density observational data to prevent over-fitting artifacts inherent in aggregated quantile fitting.

References

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